Fractional order Taylor's series and the neo-classical inequality

Keisuke Hara and Masanori Hino

Abstract

We prove the neo-classical inequality with the optimal constant, which was conjectured by T. J. Lyons [Rev. Mat. Iberoamericana 14 (1998) 215–310]. For the proof, we introduce the fractional order Taylor's series with residual terms. Their application to a particular function provides an identity that deduces the optimal neo-classical inequality.

1. *Introduction*

In his celebrated study on the theory of rough paths [**5**], T. J. Lyons introduced the following neo-classical inequality, which was a key estimate to prove one of the fundamental theorems [**5**, Theorem 2.2.1]:

Theorem 1.1 (Neo-classical inequality [**5**, Lemma 2.2.2]). *Let* $\alpha \in (0,1]$, $n \in \mathbb{N} = \{1,2,3,\dots\}$, $x \geq 0$, *and* $y \geq 0$. *Then, we have*

$$\alpha^2 \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} \leq (x+y)^{\alpha n}. \tag{1.1}$$

Here, in general, we define

$$\binom{w}{z} = \frac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)} \tag{1.2}$$

for $w \in \mathbb{C} \setminus \{-k \mid k \in \mathbb{N}\}$ and $z \in \mathbb{C}$, where $\Gamma(\cdot)$ is the Gamma function. If ∞ appears in the denominator of the right-hand side of equation (1.2), $\binom{w}{z}$ is regarded as 0.

When $\alpha = 1$, the equality holds in (1.1), which is just the conventional binomial theorem. Therefore, Theorem 1.1 is regarded as a generalisation of the binomial theorem. The proof of Theorem 1.1 in [**5**] is rather technical and is derived from the maximum principle for sub-parabolic functions. Based on the numerical evidence, Lyons conjectured that coefficient α^2 in the left-hand side of inequality (1.1) could be replaced by α. Thus far, only partial positive answers were known: E. R. Love [**3**, **4**] proved that the conjecture is true for $\alpha = 2^{-k}$ ($k = 1,2,3,\dots$) and that coefficient α^2 in (1.1) can be replaced by $\alpha/2$ in general. His proof is based on the duplication formula of the Gamma function; it seems difficult to use his method to provide a complete answer to the conjecture. Some detailed calculation along the lines of his method has also been carried out in [**2**].

In this paper, we provide an affirmative answer to Lyons' conjecture with more explicit information. Our method is different from the methods mentioned above; in our method, we use the basic theory of complex analysis. The answer to the conjecture is stated as follows.

2000 *Mathematics Subject Classification* 26D15 (primary), 30B10, 26A33 (secondary).

The second author is supported by a Grant-in-Aid for Young Scientists (B) (21740094).

Theorem 1.2. *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$, $x \geq 0$, and $y \geq 0$. Then, we have*

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} \leq (x+y)^{\alpha n}. \tag{1.3}$$

The equality holds if and only if $\alpha = 1$ or $x = y = 0$.

When $\alpha = 1$ or $x = y = 0$ holds, it is evident that inequality (1.3) holds with equality. Moreover, when $0 < \alpha < 1$ and only one of x and y is 0, inequality (1.3) is trivial with strict inequality. Therefore, it is sufficient to prove (1.3) with strict inequality for $\alpha \in (0,1)$, $x > 0$, and $y > 0$. We may assume that $x \leq y$ by symmetry. By dividing both sides by $y^{\alpha n}$ and letting $\lambda = x/y$, it is sufficient to prove the following: for $\alpha \in (0,1)$, $n \in \mathbb{N}$, and $0 < \lambda \leq 1$,

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} \lambda^{\alpha j} < (1+\lambda)^{\alpha n}. \tag{1.4}$$

In fact, we can prove the following identity.

Theorem 1.3 (Generalisation of the binomial theorem). *Let $\alpha \in (0,2)$, $n \in \mathbb{N}$, and $0 < \lambda \leq 1$. Then, we have*

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} \lambda^{\alpha j} = (1+\lambda)^{\alpha n} - \frac{\alpha \lambda^{\alpha} \sin \alpha\pi}{\pi} \int_0^1 t^{\alpha-1}(1-t)^{\alpha n} \times \left\{ \frac{1}{|t^{\alpha} - \lambda^{\alpha} e^{-i\alpha\pi}|^2} + \frac{\lambda^{\alpha n}}{|e^{-i\alpha\pi} - (\lambda t)^{\alpha}|^2} \right\} dt. \tag{1.5}$$

Since the right-hand side of equation (1.5) is clearly less than $(1+\lambda)^{\alpha n}$ for $\alpha \in (0,1)$, Theorem 1.3 immediately implies that inequality (1.4) is valid; therefore, Theorem 1.2 is proved. When $\alpha \in (1,2)$, the right-hand side of (1.5) is greater than $(1+\lambda)^{\alpha n}$. Therefore, in this case, the converse inequalities of (1.4) and (1.3) are known as the byproducts. Theorem 1.3 is proved by the application of the fractional order Taylor-like expansions with residual terms, which are obtained from the basic theory of complex analysis.

To show that coefficient α is the best constant, we denote the right-hand side of equation (1.5) by $(1+\lambda)^{\alpha n} - R(\alpha, n, \lambda)$. Then, for fixed $\alpha \in (0,1)$ and $\lambda \in (0,1]$, the error term $R(\alpha, n, \lambda)$ monotonically converges to 0 as $n \to \infty$ since the integrand decreases to 0 pointwisely. In this sense, the constant α in the left-hand sides of (1.3) and (1.4) is optimal. We can also prove that $R(\alpha, n, \lambda)$ is uniformly bounded in $\alpha \in (0,1]$, $n \in \mathbb{N}$, and $\lambda \in (0,1]$ (see Proposition 2.6 below).

This paper is organised as follows. In Section 2, we introduce the fractional order Taylor's series and prove Theorem 1.3. In Section 3, we discuss some generalisations of the main theorems.

2. *Fractional order Taylor's series and proof of Theorem 1.3*

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk in $\mathbb{C}$, and $\bar{D}$ its closure. Let f be a continuous function on $\bar{D}$ such that f is holomorphic in D.

For each $\xi \in \mathbb{R}$, we define

$$f^{\#}(\xi) := \int_{-1/2}^{1/2} f(e^{2\pi i x}) e^{-2\pi i x \xi}\, dx \tag{2.1}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{\xi+1}}\, dz, \tag{2.2}$$

where C denotes the oriented contour $(-1, 1) \ni t \mapsto \exp(i\pi t) \in \mathbb{C}$. In (2.2), $z^{\xi+1}$ is defined as $\exp\{(\xi + 1) \operatorname{Log} z\}$ on $\mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$, where the branch of Log is taken so that $\operatorname{Log} 1 = 0$. It should be noted that $f^{\#}(\xi)$ is a bounded function in ξ. Then, we have the fractional order Taylor-like series of f with residual terms as follows:

THEOREM 2.1. *For $0 < \alpha < 2$ and $0 < \lambda < 1$, the following identities hold:*

$$\alpha \sum_{j=0}^{\infty} f^{\#}(\alpha j) \lambda^{\alpha j} = f(\lambda) - \frac{\alpha \lambda^{\alpha} \sin \alpha\pi}{\pi} \int_0^1 \frac{t^{\alpha-1} f(-t)}{|t^{\alpha} - \lambda^{\alpha} e^{-i\alpha\pi}|^2}\, dt, \tag{2.3}$$

$$\alpha \sum_{j=-\infty}^{-1} f^{\#}(\alpha j) \lambda^{-\alpha j} = \frac{\alpha \lambda^{\alpha} \sin \alpha\pi}{\pi} \int_0^1 \frac{t^{\alpha-1} f(-t)}{|e^{-i\alpha\pi} - (\lambda t)^{\alpha}|^2}\, dt. \tag{2.4}$$

In particular, we have

$$\alpha \sum_{j=-\infty}^{\infty} f^{\#}(\alpha j) \lambda^{\alpha |j|}$$

$$= f(\lambda) - \frac{\alpha \lambda^{\alpha} (1 - \lambda^{2\alpha}) \sin \alpha\pi}{\pi} \int_0^1 \frac{t^{\alpha-1}(1 - t^{2\alpha}) f(-t)}{|(t^{\alpha} - \lambda^{\alpha} e^{-i\alpha\pi})(e^{-i\alpha\pi} - (\lambda t)^{\alpha})|^2}\, dt. \tag{2.5}$$

If, in addition,

$$\sum_{j=-\infty}^{\infty} |f^{\#}(\alpha j)| < \infty, \tag{2.6}$$

the identities above are valid for $\lambda = 1$; especially, equation (2.5) becomes

$$\alpha \sum_{j=-\infty}^{\infty} f^{\#}(\alpha j) = f(1). \tag{2.7}$$

REMARK 2.2. From expression (2.2), for $\xi \in \mathbb{Z}$, we have

$$f^{\#}(\xi) = \begin{cases} \dfrac{d^{\xi} f}{dz^{\xi}}(0) \Big/ \xi! & (\xi \geq 0), \\ 0 & (\xi < 0). \end{cases}$$

Therefore, when $\alpha = 1$, equation (2.3) is identical to the Taylor series expansion of f at $z = 0$, and equation (2.4) is reduced to $0 = 0$.

When $\xi \notin \mathbb{Z}$, $\Gamma(\xi + 1) f^{\#}(\xi)$ is regarded as a sort of 'ξ-order' fractional derivative of f at 0, which is denoted by $D^{\xi}_{z+1} f(0)$ in some literatures. (It should be noted that we can transform the contour C in (2.2) homotopically in $\bar{D} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$; however, the terminal points -1 should be fixed. This is the reason why $+1\ (= -(-1))$ is specified in the symbol D^{ξ}_{z+1}.) Fractional order Taylor's series have been considered in various frameworks with a variety of fractional order derivatives. (For example, see [**1**, **6**, **7**, **8**] and the references therein.) Theorem 2.1 is regarded as another variant. Equation (2.7) is consistent with the results obtained by Osler [**7**]. It should be noted that equation (2.7) can also be directly obtained by using Poisson's summation formula under the appropriate integrability condition on $f^{\#}$.

REMARK 2.3. A simple sufficient condition of (2.6) is

$$f \in C^2(S^1) \text{ and } f(-1) = 0, \tag{2.8}$$

where S^1 is the unit circle in $\mathbb{C}$. Indeed, by letting $h(x) = f(e^{2\pi i x})$ for $x \in [-1/2, 1/2]$, the integration by parts in equation (2.1) implies that

$$\begin{aligned} f^{\#}(\xi) &= \left[h(x) \cdot \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right]_{x=-1/2}^{x=1/2} - \int_{-1/2}^{1/2} \frac{h'(x)}{-2\pi i \xi} \left(\frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right)' dx \\ &= 0 + \frac{1}{4\pi^2 \xi^2} \left[h'(x) \cdot e^{-2\pi i \xi x} \right]_{x=-1/2}^{x=1/2} - \frac{1}{4\pi^2 \xi^2} \int_{-1/2}^{1/2} h''(x) \cdot e^{-2\pi i \xi x} \, dx \\ &= O(\xi^{-2}) \qquad (|\xi| \to \infty). \end{aligned}$$

Proof of Theorem 2.1. First, we prove equation (2.3). Since f is bounded on C, we have

$$\begin{aligned} \alpha \sum_{j=0}^{\infty} f^{\#}(\alpha j) \lambda^{\alpha j} &= \alpha \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(z)}{z^{\alpha j + 1}} \, dz \right) \lambda^{\alpha j} \\ &= \frac{\alpha}{2\pi i} \int_C f(z) \left(\sum_{j=0}^{\infty} z^{-\alpha j - 1} \lambda^{\alpha j} \right) dz \\ &= \frac{\alpha}{2\pi i} \int_C f(z) \frac{z^{\alpha - 1}}{z^\alpha - \lambda^\alpha} \, dz. \end{aligned} \tag{2.9}$$

We define

$$g(z) = \frac{\alpha}{2\pi i} \cdot f(z) \cdot \frac{z^{\alpha-1}}{z^\alpha - \lambda^\alpha}$$

and consider the contour Γ described in Figure 1. More specifically, C', Γ_1, and Γ_2 are defined as

$$\begin{aligned} &C' \colon (-1, 1) \ni t \mapsto \varepsilon \exp(-i\pi t) \in \mathbb{C}, \\ &\Gamma_1 \colon [-1, -\varepsilon] \ni t \mapsto t + i0 \in \mathbb{C}, \\ &\Gamma_2 \colon [\varepsilon, 1] \ni t \mapsto -t - i0 \in \mathbb{C} \end{aligned}$$

for $\varepsilon \in (0, \lambda)$.

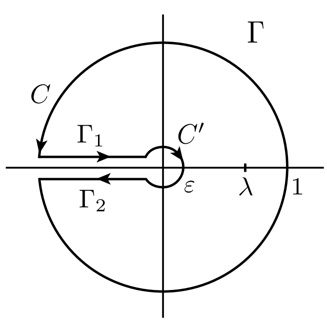

FIGURE 1. *Contour* $\Gamma = C \cup \Gamma_1 \cup C' \cup \Gamma_2$

In the domain surrounded by the contour Γ, the function g is holomorphic except at λ and has the at most first-order pole at λ. (Here, we used the assumption that $0 < \alpha < 2$.) Indeed,

we have

$$\begin{aligned}(z-\lambda)g(z) &= \frac{\alpha}{2\pi i}\cdot f(z)\cdot\frac{(z-\lambda)z^{\alpha-1}}{z^\alpha-\lambda^\alpha}\\ &\xrightarrow{z\to\lambda} \frac{\alpha}{2\pi i}\cdot f(\lambda)\cdot\frac{\lambda^{\alpha-1}}{(z^\alpha)'|_{z=\lambda}}\\ &= \frac{f(\lambda)}{2\pi i},\end{aligned}$$

and the residue of g at λ is $f(\lambda)/(2\pi i)$. From the residue theorem, we have

$$\int_\Gamma g(z)\,dz = 2\pi i\cdot\frac{f(\lambda)}{2\pi i} = f(\lambda). \tag{2.10}$$

On the circle $\{z\in\mathbb{C}\mid |z|=\varepsilon\}$,

$$|g(z)|\le\frac{\alpha}{2\pi}\cdot|f(z)|\cdot\frac{\varepsilon^{\alpha-1}}{\lambda^\alpha-\varepsilon^\alpha} = O(\varepsilon^{\alpha-1})\quad(\varepsilon\to0).$$

Therefore,

$$\left|\int_{C'} g(z)\,dz\right| \le \int_{C'}|g(z)|\,|dz| = O(\varepsilon^\alpha) = o(1)\quad(\varepsilon\to0). \tag{2.11}$$

Moreover, we have

$$\begin{aligned}\int_{\Gamma_1} g(z)\,dz &= \int_1^\varepsilon g(te^{i(\pi-0)})e^{i\pi}\,dt \qquad \text{(by the substitution } z=e^{i(\pi-0)}t)\\ &= \frac{\alpha}{2\pi i}\int_1^\varepsilon f(-t)\cdot\frac{-t^{\alpha-1}e^{i\alpha\pi}}{t^\alpha e^{i\alpha\pi}-\lambda^\alpha}\cdot(-1)\,dt\\ &= -\frac{\alpha}{2\pi i}\int_\varepsilon^1 f(-t)\frac{t^{\alpha-1}}{t^\alpha-\lambda^\alpha e^{-i\alpha\pi}}\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_2} g(z)\,dz &= \int_\varepsilon^1 g(te^{-i(\pi-0)})e^{-i\pi}\,dt \qquad \text{(by the substitution } z=e^{-i(\pi-0)}t)\\ &= \frac{\alpha}{2\pi i}\int_\varepsilon^1 f(-t)\cdot\frac{-t^{\alpha-1}e^{-i\alpha\pi}}{t^\alpha e^{-i\alpha\pi}-\lambda^\alpha}\cdot(-1)\,dt\\ &= \frac{\alpha}{2\pi i}\int_\varepsilon^1 f(-t)\frac{t^{\alpha-1}}{t^\alpha-\lambda^\alpha e^{i\alpha\pi}}\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Gamma_1\cup\Gamma_2} g(z)\,dz &= \frac{\alpha}{2\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-1}\cdot 2\,\mathrm{Re}\left[\frac{-1}{i}\cdot\frac{1}{t^\alpha-\lambda^\alpha e^{-i\alpha\pi}}\right]dt\\ &= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-1}\cdot\mathrm{Im}\left[\frac{-1}{t^\alpha-\lambda^\alpha e^{-i\alpha\pi}}\right]dt\\ &= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-1}\cdot\frac{\lambda^\alpha\sin\alpha\pi}{|t^\alpha-\lambda^\alpha e^{-i\alpha\pi}|^2}\,dt.\end{aligned} \tag{2.12}$$

Combining equations (2.9)–(2.12) and letting $\varepsilon\to0$, we obtain equation (2.3).

Equation (2.4) is proved along the same lines as the proof of equation (2.3). However, in this case, we use the following equality instead of (2.9):

$$\begin{aligned}
\alpha\sum_{j=-\infty}^{-1} f^{\#}(\alpha j)\lambda^{-\alpha j} &= \alpha\sum_{j=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{\alpha j+1}}\,dz\right)\lambda^{-\alpha j}\\
&= \frac{\alpha}{2\pi i}\int_C f(z)\left(\sum_{k=1}^{\infty} z^{\alpha k-1}\lambda^{\alpha k}\right)dz\\
&= \frac{\alpha}{2\pi i}\int_C f(z)\frac{z^{\alpha-1}\lambda^{\alpha}}{1-z^{\alpha}\lambda^{\alpha}}\,dz.
\end{aligned}$$

The integrand is holomorphic in the domain surrounded by Γ. Therefore, we have

$$\frac{\alpha}{2\pi i}\int_\Gamma f(z)\frac{z^{\alpha-1}\lambda^{\alpha}}{1-z^{\alpha}\lambda^{\alpha}}\,dz = 0.$$

Instead of equation (2.12), we consider

$$\begin{aligned}
\frac{\alpha}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} f(z)\frac{z^{\alpha-1}\lambda^{\alpha}}{1-z^{\alpha}\lambda^{\alpha}}\,dz &= \frac{\alpha}{2\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-1}\lambda^{\alpha}\cdot 2\,\mathrm{Im}\left[\frac{-1}{e^{-i\alpha\pi}-t^{\alpha}\lambda^{\alpha}}\right]dt\\
&= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-1}\lambda^{\alpha}\cdot\frac{-\sin\alpha\pi}{|e^{-i\alpha\pi}-(\lambda t)^{\alpha}|^2}\,dt.
\end{aligned}$$

The other calculations are carried out in the same manner as the proof of equation (2.3).

Equation (2.5) is simply the sum of equations (2.3) and (2.4), because

$$\begin{aligned}
&-\frac{1}{|t^{\alpha}-\lambda^{\alpha}e^{-i\alpha\pi}|^2}+\frac{1}{|e^{-i\alpha\pi}-(\lambda t)^{\alpha}|^2}\\
&=\frac{-(1-2(\lambda t)^{\alpha}\cos\alpha\pi+(\lambda t)^{2\alpha})+(t^{2\alpha}-2t^{\alpha}\lambda^{\alpha}\cos\alpha\pi+\lambda^{2\alpha})}{|t^{\alpha}-\lambda^{\alpha}e^{-i\alpha\pi}|^2|e^{-i\alpha\pi}-(\lambda t)^{\alpha}|^2}\\
&=-\frac{(1-\lambda^{2\alpha})(1-t^{2\alpha})}{|(t^{\alpha}-\lambda^{\alpha}e^{-i\alpha\pi})(e^{-i\alpha\pi}-(\lambda t)^{\alpha})|^2}.
\end{aligned}$$

When equation (2.6) holds, by taking the limit $\lambda\uparrow 1$ in equations (2.3)–(2.5), the dominated convergence theorem assures that these equations are also true for $\lambda = 1$. □

The following is another key fact for the proof of Theorem 1.3.

PROPOSITION 2.4. *Let $T>0$ and define $f(z)=(1+z)^T$ on $\bar{D}$. Then, the function $f^{\#}$ that is defined in* (2.1) *is expressed as*

$$f^{\#}(\xi) = \binom{T}{\xi}\quad \text{for } \xi\in\mathbb{R}. \tag{2.13}$$

In particular, we have

$$f^{\#}(\xi) = f^{\#}(T-\xi)\quad \text{for } \xi\in\mathbb{R}. \tag{2.14}$$

This is a classical result; for example, see [**9**] and the references therein for the proof. It should be noted that equation (2.14) is evident from expression (2.13); however, it is also directly proved by the definition of f and $f^{\#}$ and the change of variables.

Now, we prove Theorem 1.3.

Proof of Theorem 1.3. Let $0 < \alpha < 2$ and $n \in \mathbb{N}$. We define $f(z) = (1+z)^{\alpha n}$ on $\bar{D}$. First, assume that $0 < \lambda < 1$. From Proposition 2.4, we have

$$\begin{aligned}\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} \lambda^{\alpha j} &= \alpha \sum_{j=0}^{n} f^{\#}(\alpha j)\lambda^{\alpha j} \\ &= \alpha \sum_{j=0}^{\infty} f^{\#}(\alpha j)\lambda^{\alpha j} - \alpha \sum_{j=n+1}^{\infty} f^{\#}(\alpha j)\lambda^{\alpha j} \\ &= \alpha \sum_{j=0}^{\infty} f^{\#}(\alpha j)\lambda^{\alpha j} - \alpha \sum_{k=-\infty}^{-1} f^{\#}(\alpha k)\lambda^{-\alpha k}\lambda^{\alpha n}. \qquad (2.15)\end{aligned}$$

In the last equality, we substituted k for $n-j$ and used the relation $f^{\#}(\alpha n - \alpha k) = f^{\#}(\alpha k)$ that is derived from equation (2.14). Applying equations (2.3) and (2.4) in Theorem 2.1 to (2.15), we obtain the identity (1.5) in Theorem 1.3 for $\lambda \in (0,1)$. From the dominated convergence theorem, we can take the limit $\lambda \uparrow 1$ to conclude that this equation is still true for $\lambda = 1$. □

REMARK 2.5. By using the functional equality $\Gamma(z)\Gamma(1-z) = \pi/(\sin \pi z)$ and the Stirling formula $\Gamma(x)/(\sqrt{2\pi}e^{-x}x^{x-(1/2)}) \to 1$ $(x \in \mathbb{R},\ x \to +\infty)$, we can prove that

$$\binom{T}{\xi} = O(|\xi|^{-T-1}) \quad (\xi \in \mathbb{R},\ |\xi| \to \infty)$$

for $T > 0$. Therefore, the condition (2.6) holds for $f(z) = (1+z)^{\alpha n}$. If we use this fact, we do not need to make an exception the case $\lambda = 1$ in the proof of Theorem 1.3. Moreover, equation (2.7) implies that for $0 < \alpha < 2$,

$$\alpha \sum_{j=-\infty}^{\infty} \binom{\alpha n}{\alpha j} = 2^{\alpha n}.$$

This identity is a special case of more general results obtained by Osler [**7**, Eq. (5.1)].

At the end of this section, we provide a quantitative estimate of the error term $R(\alpha, n, \lambda)$ mentioned in Section 1; that is,

$$R(\alpha, n, \lambda) := \frac{\alpha \lambda^{\alpha} \sin \alpha\pi}{\pi} \int_0^1 t^{\alpha-1}(1-t)^{\alpha n} \left\{ \frac{1}{|t^{\alpha} - \lambda^{\alpha} e^{-i\alpha\pi}|^2} + \frac{\lambda^{\alpha n}}{|e^{-i\alpha\pi} - (\lambda t)^{\alpha}|^2} \right\} dt. \quad (2.16)$$

Since $R(1, n, \lambda) \equiv 0$, we can suppose that $\alpha \neq 1$.

PROPOSITION 2.6. *Let $n \in \mathbb{N}$ and $\lambda \in (0,1]$. Then, we have*

$$0 < R(\alpha, n, \lambda) < 1 - \alpha < 1 \qquad \text{for } \alpha \in (0,1)$$

and

$$0 > R(\alpha, n, \lambda) > 1 - \alpha > -1 \quad \text{for } \alpha \in (1,2).$$

Proof. First, we suppose that $0<\alpha<1$. Then, we have

$$\begin{aligned}0<R(\alpha,n,\lambda)&\le R(\alpha,1,\lambda) \qquad \text{(from (2.16))}\\ &=(1+\lambda)^\alpha-\alpha\sum_{j=0}^{1}\binom{\alpha}{\alpha j}\lambda^{\alpha j} \qquad \text{(from Theorem 1.3)}\\ &=(1+\lambda)^\alpha-\alpha(1+\lambda^\alpha)\\ &=\{(1+\lambda)^\alpha-(1+\alpha\lambda)\}+\alpha(\lambda-\lambda^\alpha)+(1-\alpha).\end{aligned}$$

Since $(1+\lambda)^\alpha$ is strictly concave in λ and its derivative at $\lambda=0$ is α, the first term in the above equation is less than 0. The second term is also dominated by 0 for $\lambda\in(0,1]$. Therefore, the above equation is less than $1-\alpha\,(<1)$.

The case that $1<\alpha<2$ is proved in the same manner; in this case, we have

$$\begin{aligned}0>R(\alpha,n,\lambda)&\ge R(\alpha,1,\lambda)\\ &=\{(1+\lambda)^\alpha-(1+\alpha\lambda)\}+\alpha(\lambda-\lambda^\alpha)+(1-\alpha).\end{aligned}$$

Since $(1+\lambda)^\alpha$ is strictly convex in λ, the first term in the above equation is greater than 0. The second term is greater than or equal to 0. Therefore, the above equation is greater than $1-\alpha\,(>-1)$. ☐

3. *Some generalisations*

In this section, we discuss some generalisations of Theorems 2.1 and 1.3. For $z\in\mathbb{C}\setminus\{0\}$, we express z as $z=re^{i\theta}$ $(r>0,\ \theta\in(-\pi,\pi])$ and define z^β as $z^\beta=r^\beta e^{i\theta\beta}$ for $\beta\in\mathbb{R}$ as a convention. The point is that the argument of z is taken in the interval $(-\pi,\pi]$.

For $\alpha>0$, we define

$$K_\alpha:=\{\omega\in\mathbb{C}\mid \omega^\alpha=1\}\,(=\{e^{i\theta}\mid -\pi<\theta\le\pi \text{ and } e^{i\theta\alpha}=1\}).$$

It should be noted that $K_\alpha=\{1\}$ when $0<\alpha<2$ and that $-1\in K_\alpha$ if and only if $\alpha/2\in\mathbb{N}$. Then, Theorems 2.1 and 1.3 are generalised as follows.

THEOREM 3.1. *Let f be a continuous function on $\bar D$ that is holomorphic in D. Let $\alpha>0$, $\gamma<\alpha$, and $0<\lambda<1$. Suppose that $\alpha/2\notin\mathbb{N}$. Then, the following identities hold:*

$$\begin{aligned}\alpha\sum_{j=0}^{\infty}f^\#(\alpha j+\gamma)\lambda^{\alpha j}&=\sum_{\omega\in K_\alpha}(\lambda\omega)^{-\gamma}f(\lambda\omega)\\ &\quad-\frac{\alpha}{\pi}\int_0^1 f(-t)t^{\alpha-\gamma-1}\frac{t^\alpha\sin\gamma\pi+\lambda^\alpha\sin(\alpha-\gamma)\pi}{|t^\alpha-\lambda^\alpha e^{-i\alpha\pi}|^2}\,dt,\end{aligned}\tag{3.1}$$

$$\alpha\sum_{j=-\infty}^{-1}f^\#(\alpha j+\gamma)\lambda^{-\alpha j}=\frac{\alpha\lambda^\alpha}{\pi}\int_0^1 f(-t)t^{\alpha-\gamma-1}\frac{\sin(\alpha-\gamma)\pi+(\lambda t)^\alpha\sin\gamma\pi}{|e^{-i\alpha\pi}-(\lambda t)^\alpha|^2}\,dt.\tag{3.2}$$

In particular, we have

$$\begin{aligned}&\alpha\sum_{j=-\infty}^{\infty}f^\#(\alpha j+\gamma)\lambda^{\alpha|j|}\\ &=\sum_{\omega\in K_\alpha}(\lambda\omega)^{-\gamma}f(\lambda\omega)-\frac{\alpha(1-\lambda^{2\alpha})}{\pi}\int_0^1 f(-t)t^{\alpha-\gamma-1}\\ &\quad\times\frac{\lambda^\alpha(1-t^{2\alpha})\sin(\alpha-\gamma)\pi+t^\alpha(1-2(\lambda t)^\alpha\cos\alpha\pi+\lambda^{2\alpha})\sin\gamma\pi}{|(t^\alpha-\lambda^\alpha e^{-i\alpha\pi})(e^{-i\alpha\pi}-(\lambda t)^\alpha)|^2}\,dt.\end{aligned}\tag{3.3}$$

If, in addition,

$$\sum_{j=-\infty}^{\infty} |f^{\#}(\alpha j+\gamma)| < \infty, \tag{3.4}$$

then, the identities (3.1)–(3.3) *are valid for* $\lambda = 1$*; in particular, equation* (3.3) *becomes*

$$\alpha \sum_{j=-\infty}^{\infty} f^{\#}(\alpha j+\gamma) = \sum_{\omega\in K_\alpha} \omega^{-\gamma} f(\omega). \tag{3.5}$$

When $\alpha/2 \in \mathbb{N}$ *and* $\gamma = 0$*, the above facts still hold by regarding the second terms of the right-hand sides of equations* (3.1) *and* (3.3) *and the right-hand side of equation* (3.2) *as zero.*

Theorem 3.2. *Let* $\alpha > 0$, $n \in \mathbb{N}$, *and* $0 < \lambda \le 1$. *Then, we have*

$$\begin{aligned}\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} \lambda^{\alpha j} = \sum_{\omega\in K_\alpha} (1+\lambda\omega)^{\alpha n} - \frac{\alpha\lambda^\alpha \sin\alpha\pi}{\pi} \int_0^1 t^{\alpha-1}(1-t)^{\alpha n} \\ \times \left\{ \frac{1}{|t^\alpha - \lambda^\alpha e^{-i\alpha\pi}|^2} + \frac{\lambda^{\alpha n}}{|e^{-i\alpha\pi} - (\lambda t)^\alpha|^2} \right\} dt.\end{aligned} \tag{3.6}$$

Here, the second term on the right-hand side of this equation is regarded as zero if $\alpha/2 \in \mathbb{N}$.
In particular, we have

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} \lambda^{\alpha j} - \sum_{\omega\in K_\alpha} (1+\lambda\omega)^{\alpha n} \begin{cases} < 0 & \text{if } 2m < \alpha < 2m+1 \text{ for some } m \in \mathbb{N}\cup\{0\}, \\ = 0 & \text{if } \alpha \in \mathbb{N}, \\ > 0 & \text{if } 2m+1 < \alpha < 2m+2 \text{ for some } m \in \mathbb{N}\cup\{0\}. \end{cases}$$

Remark 3.3.

(i) Note that $\sum_{\omega\in K_\alpha}(1+\lambda\omega)^{\alpha n}$ in equation (3.6) is a real number.

(ii) As in the case of Theorem 3.1, we can introduce the parameter γ in Theorem 3.2. However, since the introduction of γ only makes the equations complicated, we did not include it.

Proof of Theorem 3.1. The concept of this proof is the same as that of Theorem 2.1. We adopt the symbols used there. Suppose that $0 < \lambda < 1$. Then, we have

$$\begin{aligned}\alpha \sum_{j=0}^{\infty} f^{\#}(\alpha j+\gamma)\lambda^{\alpha j} &= \alpha \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(z)}{z^{\alpha j+\gamma+1}}\, dz \right) \lambda^{\alpha j} \\ &= \frac{\alpha}{2\pi i} \int_C f(z) \left(\sum_{j=0}^{\infty} z^{-\alpha j-\gamma-1} \lambda^{\alpha j} \right) dz \\ &= \frac{\alpha}{2\pi i} \int_C f(z) \frac{z^{\alpha-\gamma-1}}{z^\alpha - \lambda^\alpha}\, dz.\end{aligned} \tag{3.7}$$

Define

$$g(z) = \frac{\alpha}{2\pi i} \cdot f(z) \cdot \frac{z^{\alpha-\gamma-1}}{z^\alpha - \lambda^\alpha}.$$

Suppose that $\alpha/2 \notin \mathbb{N}$. Then, g is meromorphic in the domain surrounded by Γ, as shown in Figure 1. All the poles of g are included in $\{\lambda\omega \mid \omega \in K_\alpha\}$. In particular, the poles do not exist

on $\Gamma_1 \cup \Gamma_2$. Since for $\omega \in K_\alpha$, we have

$$\begin{aligned}(z-\lambda\omega)g(z) \xrightarrow{z\to\lambda\omega} \frac{\alpha}{2\pi i}\cdot f(\lambda\omega)\cdot \frac{(\lambda\omega)^{\alpha-\gamma-1}}{(z^\alpha)'|_{z=\lambda\omega}} \\ = \frac{(\lambda\omega)^{-\gamma}f(\lambda\omega)}{2\pi i},\end{aligned}$$

the residue of g at $\lambda\omega$ is $(\lambda\omega)^{-\gamma}f(\lambda\omega)/(2\pi i)$. From the residue theorem, we have

$$\int_\Gamma g(z)\,dz = \sum_{\omega\in K_\alpha}(\lambda\omega)^{-\gamma}f(\lambda\omega).$$

As in the proof of Theorem 2.1, we can prove that

$$\int_{C'} g(z)\,dz = O(\varepsilon^{\alpha-\gamma}) = o(1) \quad \text{as } \varepsilon\to 0$$

and

$$\begin{aligned}\int_{\Gamma_1\cup\Gamma_2} g(z)\,dz &= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-\gamma-1}\,\mathrm{Im}\left[\frac{-e^{-i\gamma\pi}}{t^\alpha-\lambda^\alpha e^{-i\alpha\pi}}\right]dt \\ &= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-\gamma-1}\cdot\frac{t^\alpha\sin\gamma\pi+\lambda^\alpha\sin(\alpha-\gamma)\pi}{|t^\alpha-\lambda^\alpha e^{-i\alpha\pi}|^2}\,dt.\end{aligned}$$

These equations imply that equation (3.1) is valid. Similarly, we have

$$\begin{aligned}\alpha\sum_{j=-\infty}^{-1} f^\#(\alpha j+\gamma)\lambda^{-\alpha j} &= \alpha\sum_{j=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{\alpha j+\gamma-1}}\,dz\right)\lambda^{-\alpha j} \\ &= \frac{\alpha}{2\pi i}\int_C f(z)\left(\sum_{k=1}^\infty z^{\alpha k-\gamma-1}\lambda^{\alpha k}\right)dz \\ &= \frac{\alpha}{2\pi i}\int_C f(z)\frac{z^{\alpha-\gamma-1}\lambda^\alpha}{1-z^\alpha\lambda^\alpha}\,dz. \qquad (3.8)\end{aligned}$$

Since the integrand is holomorphic in the domain surrounded by Γ, we have

$$\frac{\alpha}{2\pi i}\int_\Gamma f(z)\frac{z^{\alpha-\gamma-1}\lambda^\alpha}{1-z^\alpha\lambda^\alpha}\,dz = 0.$$

It also follows that the integral along C' is negligible as $\varepsilon\to 0$, and

$$\begin{aligned}\frac{\alpha}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} f(z)\frac{z^{\alpha-\gamma-1}\lambda^\alpha}{1-z^\alpha\lambda^\alpha}\,dz &= \frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-\gamma-1}\lambda^\alpha\cdot\mathrm{Im}\left[\frac{-e^{-i\gamma\pi}}{e^{-i\alpha\pi}-t^\alpha\lambda^\alpha}\right]dt \\ &= -\frac{\alpha}{\pi}\int_\varepsilon^1 f(-t)\cdot t^{\alpha-\gamma-1}\lambda^\alpha\cdot\frac{\sin(\alpha-\gamma)\pi+(\lambda t)^\alpha\sin\gamma\pi}{|e^{-i\alpha\pi}-(\lambda t)^\alpha|^2}\,dt.\end{aligned}$$

From these calculations, it is inferred that equation (3.2) holds. Equation (3.3) is obtained by adding equations (3.1) and (3.2).

Under the condition (3.4), we obtain equations (3.1)–(3.3) with $\lambda = 1$ by taking the limit $\lambda \uparrow 1$ and using the dominated convergence theorem.

When $\alpha/2 \in \mathbb{N}$ and $\gamma = 0$, the integrands on the right-hand sides of equations (3.7) and (3.8) are meromorphic in D and the poles belong to $\{\lambda\omega \mid \omega\in K_\alpha\}$. Therefore, we can directly apply the residue theorem to the integrals along C in equations (3.7) and (3.8) to obtain equations (3.1)–(3.3). □

Remark 3.4. The proof also shows that equation (3.2) holds for $\gamma < \alpha$ and $0 < \lambda < 1$, even when $\alpha/2 \in \mathbb{N}$ and $\gamma \neq 0$.

Proof of Theorem 3.2. This theorem is proved in the same way as Theorem 1.3; in this case, we use Theorem 3.1 with $\gamma = 0$ instead of Theorem 2.1. □

A table of the correspondence between some concrete functions f and $f^{\#}$ is found in, for example, [**6**, **7**] with a slightly different terminology. On the basis of this correspondence, Theorem 3.1 provides a series of nontrivial functional identities.

References

1. G. Jumarie, 'Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results', *Comput. Math. Appl.* 51 (2006) 1367–1376.
2. J.-H. Li, 'On Lyons' inequality and estimates of convergence rates of approximation via Meyer-König and Zeller operators', Master thesis in National Central University, 2004.
3. E. R. Love, 'On an inequality conjectured by T. J. Lyons', *J. Inequal. Appl.* 2 (1998) 229–233.
4. E. R. Love, 'An inequality conjectured by T. J. Lyons', Analytical methods of analysis and differential equations (Minsk, 1999), Integral Transform, *Spec. Funct.* 10 (2000) 283–288.
5. T. J. Lyons, 'Differential equations driven by rough signals', *Rev. Mat. Iberoamericana* 14 (1998) 215–310.
6. T. J. Osler, 'Leibniz rule for fractional derivatives generalized and an application to infinite series', *SIAM J. Appl. Math.* 18 (1970) 658–674.
7. T. J. Osler, 'Taylor's series generalized for fractional derivatives and applications', *SIAM J. Math. Anal.* 2 (1971) 37–48.
8. T. J. Osler, 'A correction to Leibniz rule for fractional derivatives', *SIAM J. Math. Anal.* 4 (1973) 456–459.
9. H. Pollard and O. Shisha, 'Variations on the binomial series', *Amer. Math. Monthly* 79 (1972) 495–499.

Keisuke Hara
ACCESS Co., Ltd.
Hirata Bldg.
2-8-16, Sarugaku-cho, Chiyoda-ku
Tokyo 101-0064
Japan

Keisuke.Hara@access-company.com

Masanori Hino
Graduate School of Informatics
Kyoto University
Kyoto 606-8501
Japan

hino@i.kyoto-u.ac.jp